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Notes, some key points, and minimal explication on Hans Reichenbach's "The Philosophy of Space & Time", Chapter 1: Space, (1928)
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§1

- Euclid's work established geometry as a closed and complete system.
    - The basis of [Euclidean geometry] is axioms.
    - All [Euclidean-] geometrical theorems are derived from the axioms.
    - Axiomaticism endowed [Euclidean] geometry with a certainty hitherto unsurpassed.
    - Euclidean geometry solved the "problem of demonstrability [of a science]." How? — via its axiomatic method.
- However, an epistemological question arose as a result: how is the *truth* of a given axiom justified?
    - The connection between the axioms and the theorems is *logical derivation*.
    - This logical derivation relation [between the axioms and the theorems] brought out the "implicational character" of mathematics: only *implicational statements* (such as "$a \rightarrow b$") are accessible to logical proof.
    - Euclid's axiom of the parallels (AxP) (Euclid's 5th postulate) contains a statement about *infinity*. This makes the parallel axiom the only one of Euclid's axioms for which we can't produce a single practical example.
    - Mathematicians attempted proofs for nearly two thousand years. Eventually, it was demonstrated that AxP could be dispensed with — a different axiom could be introduced and a consistent geometry developed from it. This was the advent of non-Euclidean geometry (NEG).
- How is non-Euclidean geometry (NEG) possible?
    - Logical relations compel us to accept NEG (it works out, logically speaking), and eventually NEG was proven consistent (i.e., it doesn't lead to contradictions to assume *not*-Ax P). This was done via coordination between Euclidean and NEG

axioms (i.e., if we accept Euclidean geometry, then we *must* accept NEG geometry (a contradiction in one would imply a contradiction in the other)).

- As a result of these discoveries, a plurality of geometries became recognized. The axioms of geometry are *not* true or false — they are arbitrary statements.
- This raises a new question: into which discipline should the truth of a given axiom *a* be incorporated?
    - This is a physical question — *not* a mathematical one: it divides space into mathematical and physical.
- Physics decides which among revealed mathematical possibilities corresponds to reality.
    - Which of the plurality of geometries corresponds to reality?
    - A question thus arises: what *methods* should physics employ to come to a decision? (This question is picked up again in §3.)

§2

- Riemann's [extension of the] concept of space centered around the concept of 'metric' (a standard of measurement).
    - Riemann developed Gauss's result that the shape of a curved surface can be characterized by the geometry *within* (as opposed to outside) the surface (i.e. as opposed to being characterized by a point's deviation from a plane).
    - Briefly: This is done via taking practical measurements on the surface and noting a deviation [in ratio between diameter and circumference] from $\boldsymbol{\pi}$.
    - This shows that different metrical relations hold for surface geometry than plane geometry.
- Riemann's [extended] definition of the concept of space includes Euclidean geometry and Lobatschewsky's geometry as special cases.
- Thus: space is a 3D manifold (the question is left open: which axiomatic systems will hold for it?).

§3

- The method physics should employ [to come to a decision about which of the plurality of geometries corresponds to reality] should be brought about by practical measurements in space (as in Riemann's procedure).

Thought Experiment: Projection of NEG on a plane

Let:

- Surface G: solid, see-through hemisphere — plane except for a hump in the middle.
- Surface E: solid, opaque plane below and parallel to G. Light casts shadows of objects on G onto E.
- People live on G (G-people) and E (E-people).
- G people discover via measurements (as in Riemann) that G is non-Euclidean.
- They measure the distances $A^G B^G$ and $B^G C^G$ as equal in length. Note: the distances of the corresponding shadows on E — $A^E B^E$ and $B^E C^E$ — would be called "unequal in length."
- E has a mysterious force that varies the length of all measuring rods moved on it such that they are always equal in length to the corresponding shadows [cast from G]. Note: This change cannot, thus, be directly perceived by E-people (all other objects are also affected).
    - The E-people would thus obtain the same measurement results as the G-people.

- Then, on the mysterious force: how can the force be detected if the nature of the geometry *cannot* be used as an indicator?
- Two properties this force and all [what will be called] "universal" forces have: (a) they affect all materials in the same way; and, (b) there are no insulting walls (coverings which don't act on the enclosed object with forces having (a)).
    - Def. Universal forces = (a) + (b) forces.

  - Def. Differential forces = All forces ¬universal forces.
    - Only differential forces are directly demonstrable.
- We've overlooked an assumption: the determination of geometry depends on the question whether or not two distances are really equal in length, but to know if two distances are really equal, we have to know beforehand what it means to say that two distances are 'really equal'. (The assumption here is about what 'really equal' means — i.e., its definition.)

§4

- A "definition" usually involves reducing a concept to another concept. A second kind of definition: concepts are coordinated with real objects.
  - In general, this coordination is *not* arbitrary. The concepts are interconnect by testable relations (The length can be tested against another length). Thus, the coordination may be verified as truth or false (if it is unique).
  - Preliminary coordinations must be made first. These are coordinative definitions.
  - Coordinative definitions are arbitrary (i.e., there's no initial normative principal at work; what you pick doesn't matter—given certain constraints).
    - The method: coordination of a concept to a physical object.
    - Example: if a distance has to be measured, the *unit of length* must be determined beforehand by definition. This is a coordinative definition.
- On the comparison of two units of length at different locations: if a measuring rod is laid down, its length is compared only to that part of a body which it covers at the moment. It's assumed the rod doesn't change if transported. This is impossible to know. Thus, a coordinative definition is used: two objects are defined as equal in length (congruence).
- The determination of the geometry of a certain structure depends on the definition of congruence (i.e. on coordinative definition) (pp. 18).
  - Whether $A^E B^E = B^E C^E$ is a matter of definition. The geometrical form of a body depends on a preceding coordinative definition.
- New problem: which coordinative definition should be used for physical space? The geometry of physical space depends on this definition.

- We'll look for one with (a) the most logical simplicity, and that (b) requires the least possible change in the results of science.

§5

- No object is the perfect realization of a solid body in physics.
- Rigid body: Those bodies that constitute the physical part in the coordinative definition of congruence and that do not change size when transported (by definition) (pp. 22).
- Rigid bodies are solid bodies unaffected by differential forces. We set the universal forces equal to 0 by definition.
- Conservation of shape: Lack of exterior forces. Change of shape is called small if the exterior forces are small relative to the interior forces (pp. 23).

§6

- Elaboration of the difference between universal and differential forces (for ease: the basic definitions are repeated below).
    - Def. Universal forces = (a) affect all materials in the same way + (b) have no insulting walls (coverings which don't act on the enclosed object with forces having (a)).
    - Def. Differential forces = All forces ¬universal forces.

§7

- Technical possibility versus logical possibility. Without a coordinative definition, the question of whether E is a plane or a surface with a hump *cannot* be decided (there's no limit any measurement could approach). Once a coordinative definition has been given, the technical impossibility still remains. All our measurements will contain some degree of technical inexactness.

§8

- We've decided that the question of which geometry holds for physical space must be decided by measurements (that is, empirically). In addition, this decision is dependent on the assumption of an arbitrary coordinative definition of the comparison of length.
- *Theorem 0*: "Given a geometry *G′* to which the measuring instruments conform, we can imagine a universal force *F* which affects the instruments in such a way that the actual geometry is an arbitrary geometry *G*, while the observed deviation from *G* is due to a universal deformation of the measuring instruments" (pp. 23).
- *The principle of the relativity of geometry*: [as theorem 0 shows] all geometries are equivalent. No one particular geometry is the "true" geometry.
    - We obtain a statement about physical reality only if in addition to its geometry *G* its universal field of force *F* is specified — thus, only $G + F$ is testable.
    - One argument states Euclidean geometry has the simpler geometry and so physics must choose the coordinative definition $G = G_0$ (rather than $F = 0$).
        - To answer this: physics doesn't care which *geometry* is simpler — it cares which *definition* is simpler. And that is the definition $F = 0$, as $G + F = G$.
        - We can't say Einstein's NEG is "truer" than Euclidean geometry — but we can say it is *simpler* (it is simpler because in it $G + F = G$). It is a *convention* to set $F = 0$ — and its *descriptive simplicity* has nothing to do with *truth* — but the simpler system is always preferable.
- The principle of the relativity of geometry *does not* mean geometrical statements about objective reality become arbitrary. Simply because we define the scale of temperature does *not* mean the indication of the temperature of an object becomes a subjective matter.
    - In fact, objective physical measuring results are only determined *because* we classify points of arbitrariness as arbitrary (otherwise the results are *undetermined*, as we don't know how the metrical system actually functions).

§9 - §11

(Discusses Euclidean and non-Euclidean geometrical visualization.)

§12

Discussion of Torus Space

- Curves on a plane can be contracted to a point — this is *not* so for [every curve on] a torus (pp. 60). Thus, on a torus, a "between" relation (i.e. $B(x,y,z)$) is *undetermined*. This indeterminacy means such a curve doesn't divide the surface into two separate domains. The "hole" of the torus is a matter of a third dimension — the surface itself has no such hole, and thus, walking on the surface is an uninterrupted environment. For reference: Figure 8 is used in this example (pp. 64).
- Call a measurer person on such a surface P. P assumes space is Euclidean. P measures 1 as a spherical shell. P moves to 2 and finds that 2 is smaller than 1—indicating that 2 is inside 1. Further on, P finds 3 is as large as 1— however, 3 *should be* smaller than 1 (and 2) by Euclidean space. P explains this by appeal to a universal force. Next, P finds 4 is larger than 3 (thus, also larger than 1 and 2); and, that 5 is as large as 1 and 3. What's "stranger" at 5, however, is that P finds that everything at 5 also turns out to be at 1 (i.e. the room, the drawer, etc.).
    - To condense what P has observed thus far:
        - Shells 1, 3, and 5 appear to be the same size.
        - Shell 2 is smaller than shells 1, 3-5.
        - Shell 4 is larger than shells 1-3, 5.
        - The same things are encountered on shell 1 and shell 5.
- P is certain that surface 1 must be separated from surface 5. This is because P relies on a betweenness relation for observations of these surfaces (e.g. checking that *Between*(2,1,3)). If this is the case, however, P needs to explain how they can wander from 1 (outside direction) or 5 (inside direction) and yet encounter the same things. This explanation can't be that *normal causality* holds — since the causality described doesn't

require (a) time for transference; and, (b) doesn't spread as a continuous effect passing consecutively through the intermediate points. Thus, [according to P] there must be — in addition to the universal force — a *causal* anomaly.

- So the discussion implies something like this: "*If* it is assumed that every space is Euclidean, *then* there must be a causal anomaly [at least in some spaces]."
    - This can be reformulated in the way closer to what Reichenbach later relies on: "*If* it is assumed that every space is Euclidean, *then* [normal causality] does *not* hold [in at least some spaces]." To assume the antecedent here (as the Kantian philosophy goes), entails normal causality *doesn't* hold, and that's counter to the *a priori* principle (which says it does).
- This leads to "…the strongest refutation of the philosophy of the *a priori*" argument (pp. 67).

Argument close-to-text:

1. The aprioristic philosopher *cannot* be prevented from retaining Euclidean geometry (from the relativity of geometry).
2. *If* the aprioristic philosopher retains Euclidean geometry, *then* he must renounce normal causality as a general principle (from torus space analysis).
3. According to the aprioristic philosopher, normal causality is an *a priori* principle.
4. Thus, he must renounce *either* Euclidean geometry *or* causality.
5. However, in such a case, we deal with perceptions which no *a priori* principle could change.
6. Thus, there are conceivable circumstances under which two *a priori* requirements postulated by philosophy would contradict each other.

Argument reformulated for logical structure:

1. If EG is the geometry of space, then [normal] causality does *not* always hold (supported by the torus space analysis (pp. ~58-66)). Premise 1: $A \rightarrow \neg B$
    2. Assume for reductio: Euclidean geometry (EG) is the geometry of space (*a priori*); and, [normal] causality always holds (*a priori*). [Assume:] $A \wedge B$

3. EG is the geometry of space (from 2; conjunction elimination). A
4. So, causality does *not* hold (from 1 and 3; conditional elimination). ¬B
5. But causality *does* hold (from 2; conjunction elimination). B
6. So, causality both does and does *not* hold (from 4 and 5; contradiction introduction). ⊥

7. Thus, 2 is false (¬2) (from 2-6; negation introduction). ¬(A ∧ B)

§13

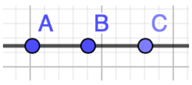
(Further discussion on visualization.)

§14

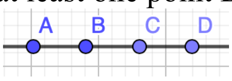
Discussion of Hilbert's "between" axioms (pp. 94 - 95) including updated logical calculus

- The three "*between*" axioms (i.e. the axioms which define the ternary *between*):

1. If A, B, and C are points on a straight line, and B lies between A and C, then B lies between C and A.



2. If A and C are two points on a straight line, there exists at least one point B which lies between A and C and at least one point D such that C lies between A and D



3. Among any three points which lie on a straight line there exists exactly one which lies between the other two.

Logical symbols

∧	and (conjunction)
∨	or (disjunction)
→	implies (has the consequence)
$\forall x$	universal quantification (for all x…)
$\exists x$	existential quantification (for some x; there is at least one x such that…)
$\neg x$	negation (it is *not* the case that x)

Geometrical symbols

$point(x)$	x is a point
$line(x)$	x is a straight line

$lies(x, y)$	x lies on y
$between$	x is between y and z

- The three "*between*" axioms now read:

(1) $\forall xyz(point(x) \wedge point(y) \wedge point(z) \wedge \exists w(line(w) \wedge lies(x,w) \wedge lies(y,w) \wedge lies(z,w)) \wedge (between(y,x,z) \rightarrow between(y,z,x)))$

∀xyz[
point(x) ∧ point(y) ∧ point(z) ∧
∃w(line(w) ∧ lies(x,w) ∧ lies(y,w) ∧ lies(z,w)) ∧
(between(y,x,z) → between(y,z,x))
]

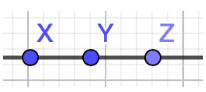
(2) $\forall xz(point(x) \wedge point(z) \wedge (x \neq z) \wedge (\exists w(line(w) \wedge lies(x,w) \wedge lies(z,w)) \rightarrow (\exists y(point(y) \wedge between(y,x,z)) \wedge \exists v(point(v) \wedge between(z,x,v))))$

∀xz[
point(x) ∧ point(z) ∧ (x≠z) ∧
(∃w(line(w) ∧ lies(x,w) ∧ lies(z,w)) → (∃y(point(y) ∧ between(y,x,z)) ∧
∃v(point(v) ∧ between(z,x,v)))
]

(3) $\forall xyz(point(x) \wedge point(y) \wedge point(z) \wedge (\exists w(line(w) \wedge lies(x,w) \wedge lies(y,w) \wedge lies(z,w) \rightarrow (between(y,x,z) \vee between(x,y,z) \vee between(z,x,y)) \wedge \neg(between(y,x,z) \wedge between(x,y,z)) \wedge \neg(between(y,x,z) \wedge between(z,x,y)) \wedge \neg(between(x,y,z) \wedge between(z,x,y)))$

∀xyz[
point(x) ∧ point(y) ∧ point(z) ∧
(∃w(line(w) ∧ lies(x,w) ∧ lies(y,w) ∧ lies(z,w)) →
(between(y,x,z) ∨ between(x,y,z) ∨ between(z,x,y)) ∧
¬(between(y,x,z) ∧ between(x,y,z)) ∧ ¬(between(y,x,z) ∧ between(z,x,y)) ∧
¬(between(x,y,z) ∧ between(z,x,y))
]

- "…Only the logical symbols have an independent meaning; the geometrical symbols have a derived meaning…" in that they denote elements and relations to satisfy axioms.
- Demonstration that the axioms entail a restriction such that only a certain number of the visual relations of geometry satisfy the axioms:



- Which point lies *between* the other two?
    - Suppose it is *z*.

- Then we're interpreting $between$ as the visual relation "lies outside."
- Axiom 1 and 2 are compatible with this interpretation, but 3 is not.
- 3 is not compatible because — for z to satisfy $between$ as "lies outside" — it would require that $between(x, y, z) \wedge between(z, x, y)$ (i.e. that x also satisfy this symbolic expression. However, axiom 3 states $between(x, y, z) \wedge between(z, x, y)$ must be false.
- Thus, "…only the visual relation *between* and *not* the visual relation *lies outside* satisfies the relation $between$ which was defined exclusively by logical symbols."

"The visual elements cannot be exhaustively defined by the basic logical concepts alone," (pp. 96).

§15

(Discusses graphical representation.)